TITLE

Overview of my research between 2004 and 2007

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Story of the quest for the one ring to rule them all

ALGEBRA BACKGROUND

In algebra one deals with numbers.	**False!** Algebra studies any set equipped with any operations. The elements do not have to be numbers. **Elementary vs. Abstract.**
Algebra is what one studies in high school - there is no serious research in algebra.	**False!** There are many nontrivial questions that are still open in algebra.
Algebra does not relate to other fields.	**False!** Algebra is widely used in various other disciplines (e.g. chemistry, physics, and many other).

ALGEBRA BACKGROUND

Algebra is a branch of mathematics concerning the study of structure, relation and quantity.

Abstract algebra extends the familiar concepts found in elementary algebra and arithmetic of numbers to more general concepts

SETS equipped with OPERATIONS

that satisfy certain laws.

WHAT IS A RING?

A **ring** is a set R with two binary operations + and · (addition and multiplication) satisfying the following **RING AXIOMS.** for all a,b,c in R:

1. $(a+b)+c = a+(b+c)$
2. $a+0=0+a=a$
3. $a+(-a) = (-a)+a = 0$
4. $a+b=b+a$
5. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
6. $a \cdot (b+c) = a \cdot b + a \cdot c$
   $(b+c) \cdot a = b \cdot a + c \cdot a$.
7. If $a \cdot 1 = 1 \cdot a = a$
   then the ring R is called a **ring with identity**.

DIVIDING IN A RING?

Problem: You cannot divide in every ring!

For example, there are no integer solutions of $2 \cdot x = 1$.

If
$ax=1$ and $ya=1$
have solutions
for
every non-zero a
in R

⇒

the ring R is
called a
division ring.

Goal: embed any non-division ring in a division ring.

Familiar example: Extend the integers Z to rational numbers Q by introducing fractions. Embedding: $k \rightarrow k/1$.

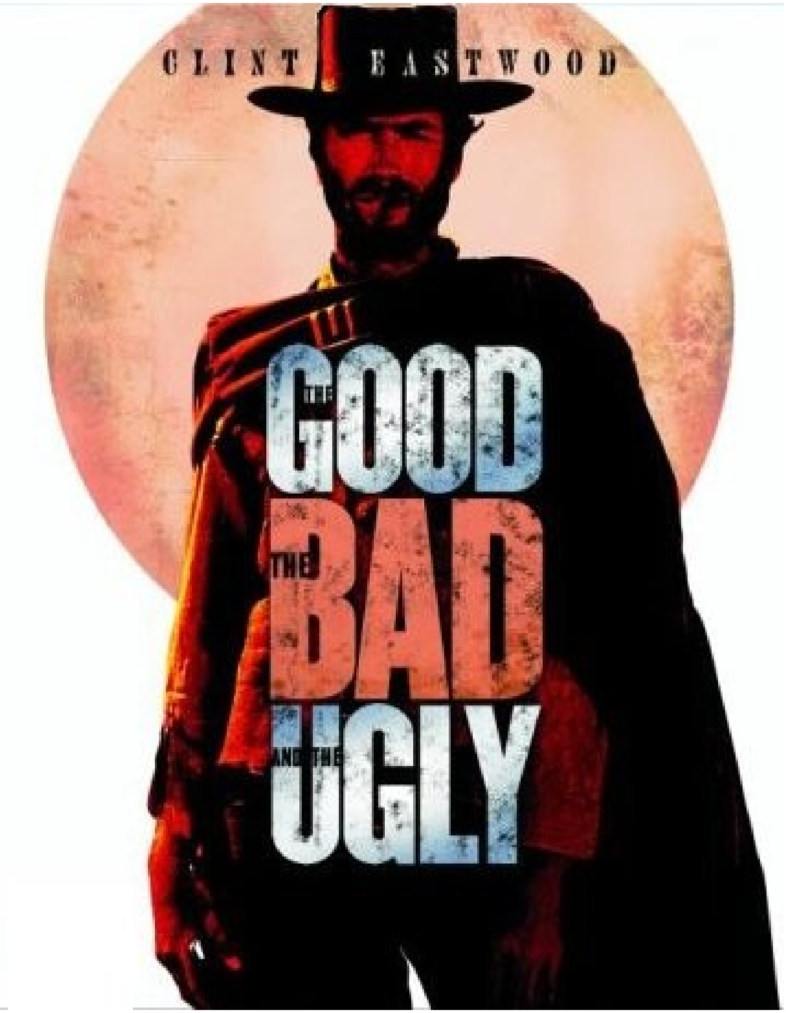
AND SO THE QUEST BEGINS...

<u>Noncommutative localization</u>
as described by T. Y. Lam:

- **“The Good”** -- some rings can be embedded in division rings;

- **“The Bad”** -- not all rings can;

- **“The Ugly”** -- even those that can might have nonisomorphic division rings of fractions.



SEARCHING...

In the work of Lambek, Utumi, and others in 1950s and early 1960s, a **”nonclassical”** ring of quotients emerged.
$Q^r_{max}(R)$ – **the maximal ring of quotients**.

- **“The Good”** -- every ring R has it.
- **“The Bad”** -- you cannot exactly divide in it.
- **”The Ugly”** -- often it is too large.

So, the search for other types of right rings of quotients continued...

FINDING?

In the work of Findlay, Morita, Lazard, and others in late 1960s and early 1970s, another **nonclassical** ring of quotient emerged.

$Q^r_{tot}(R)$ – **the total ring of quotients**.

It seemed that $Q^r_{tot}(R)$ was “just right”.

$Q^r_{cl}(R) \leq Q^r_{tot}(R) \leq Q^r_{max}(R)$

PERFECT QUOTIENT

The usual construction of $Q^r_{tot}(R)$	Start with R build up on it till you reach $Q^r_{tot}(R)$.
Morita's construction of $Q^r_{tot}(R)$	Start with $Q^r_{max}(R)$, shrink it down till you reach $Q^r_{tot}(R)$.

Problem: The usual construction is not explicit. Morita's construction takes infinitely many steps.

MY IMPROVEMENTS OF THE CONSTRUCTION

In 2005, I improved Morita's construction:

“The Good”	Easier, more hands-on description of $Q^r_{tot}(R)$. Construction ends after just one step for some Rs.
“The Bad”	Can be done just under a certain condition.
“The Not-So-Ugly”	Condition holds for some important classes of rings.

MY WORK IN 2007

“The Good”	Defining the $Q^{r}_{tot}(R)$ as **double**-sided fractions. Takes care of **commutativity**. Example of non-commutativity: $2(x+3) \neq 2x+3$ In a non-commutative ring $ab^{-1} \neq b^{-1}a$ and we would like to have both left and right fractions.
Not Bad	
Not Ugly	

AND NOW FOR SOMETHING COMPLETELY DIFFERENT

Von Neumann algebras. Used in: geometry, probability, statistical mechanics, representation theory, topology etc.

Named after John von Neumann	⇨	He wanted to capture abstractly the concept of an algebra of observables in quantum mechanics.
Limits of Hilbert space probability theory:	⇨	not capable of describing large (infinite in size or in degrees of freedom) quantum systems.
VNA has a <u>dimension function</u>	⇨	Corresponds to normalized measure in classical probability space.

FAMOUS QUOTE

"Von Neumann algebras are blessed with an excess of structure -- algebraic, geometric, topological -- so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work. (...) If all the functional analysis is stripped away (...) what remains should (be) completely accessible through algebraic avenues". -- S.K. Berberian, "Baer *-rings", Springer-Verlag, Berlin-Heidelberg-New York, 1972.

In other words:

VNAs have a rich structure. So rich that it is hard to study them.

"SCIENCE DIRECT PAPER", 2004/2005

- I generalized many known results on finite von Neumann to the class **C**.

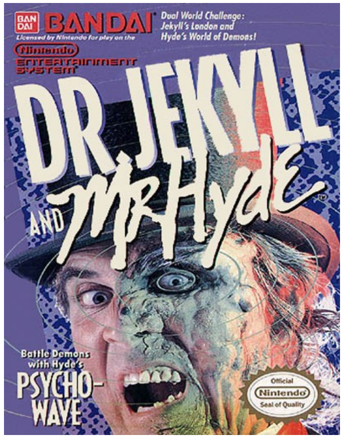
2. I defined

the dimension

for the class **C** and proved that it has all the nice properties of the dimension for finite von Neumann algebras.

3. I simplified the axioms used to define the class **C**. This provided an **answer to a question** posed by Berberian that remained open since the 70s.

SPLIT PERSONALLITY



Real functions	Rings, Modules
Derivatives	Rings of Quotients
Integrals	Torsion theories
Mutlivariable Calculus	Baer *-rings
Analysis	**Algebra**

DERIVATIVES IN A RING!

Ring R has **derivation** *d* if $d: R \to R$ and for all a,b in R

$$d(a+b)=d(a)+d(b)$$
$$d(ab)=d(a)b+ad(b)$$

Note that this is an "artificial", not intrinsically a ring theoretic notion.

Want: it agrees with other ring theoretic notions.

IT WORKS!

For example, if you know *d* on R (or a module M), you would want to be able to extend it on its rings and modules of quotients.

1. $Q^r_{cl}(M)$ -- can be done (known before)

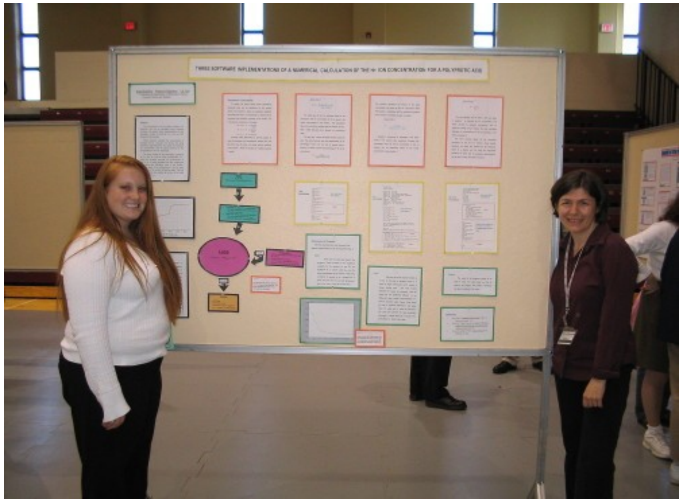
2. $Q^r_{max}(M)$ -- can be done (Vas, 2006)

3. $Q^r_{tot}(M)$ -- can be done (Vas, 2006)

UNDERGRADUATE RESEARCH

Undergraduate research Math/Chemistry -- **baby steps**

In 2005, with K. Beishline and R. Napoleon - simplifying calculations of the hydrogen ion concentration in polyprotic acids



In 2006, with 4 chem/bio.chem students. Talks based on ideas to be included in MA360, Topics in Mathematics with Applications in Chemistry

IF YOU HAVE AN IDEA FOR COLLABORATION...

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