Total Right and Symmetric Rings of Quotients

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Thank you, Ohio!
Happy birthday, S. K. Jain !

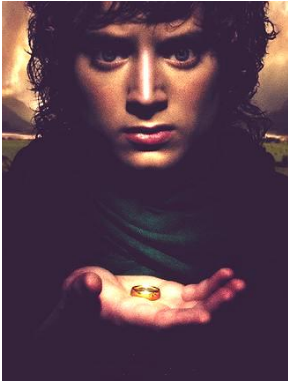
Motivation

Noncommutative localization as described by T. Y. Lam:

- **”The Good“** – some rings can be embedded in division rings;
- **”The Bad“** – not all rings can;
- **”The Ugly“** – even those that can might have nonisomorphic ”division rings of fractions“.

Continuing the search...

In the work of Lambek, Utumi, Findlay and others from the 1950s and early 1960s, the idea of a "nonclassical" ring of quotients emerged.



- **"The Good"** – every ring R has the maximal right ring of quotients $Q^r_{\mathrm{max}}(R)$.
- **"The Bad"** – you cannot exactly divide in Q^r_{max} in many cases.
- **"The Ugly"** – Q^r_{max} is the largest of the general right rings of quotients. Often it is too large.

Torsion theories – unified treatment of different rings of quotients

A **torsion theory** for a ring R is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of R-modules such that

i) $\mathrm{Hom}_R(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

ii) $\mathcal{T}$ and $\mathcal{F}$ are maximal classes with property i).

$\mathcal{F}$ is closed under taking	$\mathcal{T}$ is closed under taking
submodules products extensions	quotients sums extensions
	submodules (then τ is **hereditary**)

Background on torsion theories (cont.)

For every R-module M this gives us:

1. **Torsion submodule** $\mathcal{T}M$, largest submodule that belongs to $\mathcal{T}$,
2. **Torsion-free quotient** $\mathcal{F}M = M/\mathcal{T}M$,
3. **Filter** $\mathfrak{F} = \{ \; I \text{ right ideal } \mid R/I \in \mathcal{T} \; \}$.
4. **Right ring of quotients** with respect to hereditary τ

$$Q^r_{\mathfrak{F}}(R) = \varinjlim_{I \in \mathfrak{F}} \mathrm{Hom}_R(I, \mathcal{F}R)$$

5. **Right module of quotients** with respect to hereditary τ

$$Q^r_{\mathfrak{F}}(M) = \varinjlim_{I \in \mathfrak{F}} \mathrm{Hom}_R(I, \mathcal{F}M)$$

Perfect quotients and filters

If

$$Q^r_{\mathfrak{F}}(M) \cong M \otimes_R Q^r_{\mathfrak{F}}(R)$$

for every M, then a module of quotients is determined solely by the ring $Q^r_{\mathfrak{F}}(R)$.

In this case

- The filter $\mathfrak{F}$ is called **perfect** and
- $Q^r_{\mathfrak{F}}(R)$ is called the **perfect right ring of quotients.**

The classical torsion theory of a right Ore ring is perfect so, a perfect torsion theory is a way to generalize the classical torsion theory in cases when the ring might not be right Ore.

The ”most perfect“ quotient

In the late 1960s and early 1970s, Findlay, Knight, Lazard, Popescu and Spircu considered a ring of quotients called:

the total right ring of quotients $Q^r_{\mathrm{tot}}(R)$.

$Q^r_{\mathrm{tot}}(R)$ is the largest perfect quotient that contains R. It is **”good“** – exists for every ring. As

$$Q^r_{\mathrm{cl}} \subseteq Q^r_{\mathrm{tot}} \subseteq Q^r_{\mathrm{max}}$$

it seemed that $Q^r_{\mathrm{tot}}(R)$ was **”just right”**.

Why else is Q^r_{tot} "The Good"?

In Q^r_{tot}, for every element a

$a = \sum_i s_i \overline{t_i}$	$s_i = at_i \in R,\ t_i \in R$	$\sum_i t_i \overline{t_i} = 1$

Compare with property for every $a \in Q^r_{\mathrm{cl}}$

$a = s\overline{t}$	$s = at \in R, \quad t \in R$	$t\overline{t} = 1$

The usual and Morita's constructions of Q^r_{tot}

Usual Construction: The directed union of all perfect quotients that extend R is Q^r_{tot}.

- Start from R,
- Go upwards towards $Q^r_{\max}$, build up on R.

Morita's idea:

- Start from $Q^r_{\max}$,
- Go downwards towards R, shrink $Q^r_{\max}$ down.

Descending is done by transfinite induction.

Simplification of Morita's construction

All rings constructed inductively in Morita's construction are rings of right quotients of a certain torsion theory.

Simplification of the construction

=

simplification of the description of the torsion theory.

The Good: Easier, more hands-on description of Q^r_{tot}.

The Bad: Can be done just under a certain condition (C).

The Not-So-Ugly: Condition (C) holds for some important classes of rings including right semihereditary rings.

Q^r_{tot} for R right semihereditary

Theorem. If R is right semihereditary, then Morita's construction ends after one step and Q^r_{tot} is the ring of quotients with respect to $\mathfrak{F} = \{\, I \mid IQ^r_{\max} = Q^r_{\max} \,\}$. Thus

$$Q^r_{\text{tot}} = \{q \in Q^r_{\max} \mid qI \subseteq R \text{ for some } I \text{ with } IQ^r_{\max} = Q^r_{\max}\} =$$
$$\{q \in Q^r_{\max} | qr_i \in R \text{ for some } r_i \in R, s_i \in Q^r_{\max} \text{ s.t. } \sum r_i s_i = 1\}$$

The proof follows from 3 facts:

1. If α is successor, the α-th filter has a basis of finitely generated right ideals.
2. If R is right semihereditary, those ideals are projective.
3. Every filter with a basis of finitely generated projective ideals is perfect.

Right semihereditary rings – cont.

Right semihereditary rings with $\boxed{Q^r_{\max} = Q^l_{\text{tot}}}$ have been studied in the past (Goodearl, Cateforis, Evans, Finkel Jones). For those rings $\boxed{\text{fin. gen. nonsingular} \quad = \quad \text{projective.}}$

Open Problem: Stenström is asking for necessary and sufficient conditions for $\boxed{Q^r_{\max} = Q^r_{\text{tot}}}$ to be equal.

Note: There are examples of right semihereditary rings R with:

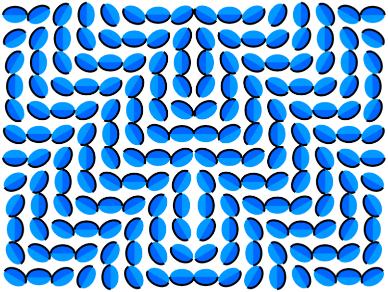
1. $Q^r_{\max} = Q^r_{\text{tot}}$, Lambek t.t perfect.
2. $Q^r_{\max} = Q^r_{\text{tot}}$, Lambek t.t not perfect.
3. $Q^r_{\max} \neq Q^r_{\text{tot}}$.

Symmetry - also “good”

Non symmetry - another “badness”. **Solution: symmetric rings of quotients**.

- symmetric Martindale rings of quotients,
- Symmetric version of $Q^r_{\max}$ called **the maximal symmetric ring of quotient.** Studied by Lanning in 1990s and Ortega in 2000s.

Schelter's (1970s) – work on symmetric rings of quotients that parallels Gabriel's work on right rings of quotients.
Ortega (2000s) – symmetric modules of quotients.



Symmetric filters and torsion theories

Schelter's idea.

$$\text{Symmetric for } R = \text{Right for } R \otimes_{\mathbb{Z}} R^{op}.$$

Symmetric Filter: Start with $\mathfrak{F}_l$ left and $\mathfrak{F}_r$ right filters. Define $\mathfrak{F}$ = right ideals of $R \otimes_{\mathbb{Z}} R^{op}$ containing an ideal of the form

$$J \otimes R^{op} + R \otimes I$$

$I \in \mathfrak{F}_l$ and $J \in \mathfrak{F}_r$. Corresponds to torsion theory τ for which

$$\mathcal{T}(M) = \mathcal{T}_l(M) \cap \mathcal{T}_r(M).$$

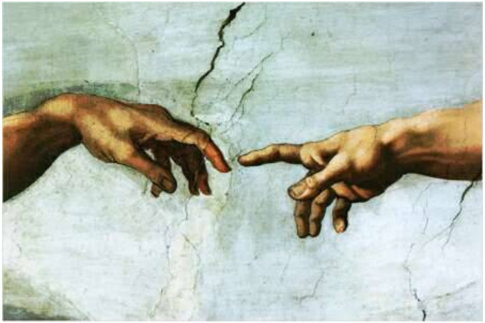
Ortega – symmetric modules of quotients:

$$Q^{\sigma}_{\mathfrak{F}}(M) = \varinjlim_{K \in \mathfrak{F}} \mathrm{Hom}(K, \tfrac{M}{\mathcal{T}(M)})$$

Perfect and symmetric combined – two “goods” interacting

To do list. Define and study:

1. Symmetric version of perfect right rings of quotients.
2. Symmetric version of perfect right filters.
3. Symmetric version of the total right ring of quotients Q^r_{tot}.



Perfect rings of quotients – right vs symmetric

$f : R \longrightarrow S$ ring homomorphism. For S to be

perfect right	[Vaš] **perfect symmetric**

ring of quotients, any of the following equivalent conditions needs to hold.

f is a ring epi S is left R-flat	f is a ring epi S is left and right R-flat
$S \cong Q^r_{\mathfrak{F}}$ for $\mathfrak{F}_r = \{J \mid f(J)S = S\}$	$S \cong Q^\sigma_{\mathfrak{F}}$ for $\mathfrak{F}$ induced by $\{J \mid f(J)S = S\}$ and $\{I \mid Sf(I) = S\}$

Perfect filters – right vs symmetric

Let $\mathfrak{F}_r$ be right, $\mathfrak{F}_l$ left, $\mathfrak{F}$ induced symmetric filter, q^r the localization $q^r : R \longrightarrow Q^r_{\mathfrak{F}_r}$ and q^σ the localization $q^\sigma : R \longrightarrow Q^\sigma_{\mathfrak{F}}$. Then

	[Vaš]
$\mathfrak{F}_r$ is perfect:	**$\mathfrak{F}$ is perfect:**
$Q^r_{\mathfrak{F}_r}$ is perfect and $\mathfrak{F}_r = \{J \mid q^r(J)Q^r_{\mathfrak{F}_r} = Q^r_{\mathfrak{F}_r}\}$	$Q^\sigma_{\mathfrak{F}}$ is perfect and $\mathfrak{F}_l = \{I \mid Q^\sigma_{\mathfrak{F}} q^\sigma(I) = Q^\sigma_{\mathfrak{F}}\}$ $\mathfrak{F}_r = \{J \mid q^\sigma(J) Q^\sigma_{\mathfrak{F}} = Q^\sigma_{\mathfrak{F}}\}$
$M \otimes Q^r_{\mathfrak{F}_r} \cong Q^r_{\mathfrak{F}_r}(M)$	$Q^\sigma_{\mathfrak{F}} \otimes M \otimes Q^\sigma_{\mathfrak{F}} \cong Q^\sigma_{\mathfrak{F}}(M)$

Symmetric version of Q^r_{tot}

$Q^r_{\mathrm{tot}}(R)$ **is defined as**	[Vaš] $Q^\sigma_{\mathrm{tot}}(R)$ **is defined as**
direct limit (maximal element) of the directed family of all perfect **right** rings of quotients of R.	direct limit (maximal element) of the directed family of all perfect **symmetric** rings of quotients of R.

Properties:

$Q^r_{\mathrm{tot}} \subseteq Q^r_{\mathrm{max}}$	$Q^\sigma_{\mathrm{tot}} \subseteq Q^\sigma_{\mathrm{max}}$
R right hereditary and right noetherian $Q^r_{\mathrm{tot}} = Q^r_{\mathrm{max}}$	R hereditary and noetherian $Q^\sigma_{\mathrm{tot}} = Q^\sigma_{\mathrm{max}} = Q^r_{\mathrm{max}}$
R right Ore $Q^r_{\mathrm{cl}} \subseteq Q^r_{\mathrm{tot}} \subseteq Q^r_{\mathrm{max}}$	R left and right Ore $Q_{\mathrm{cl}} \subseteq Q^\sigma_{\mathrm{tot}} \subseteq Q^\sigma_{\mathrm{max}}$

Examples

Vaš. Examples with

$R \subsetneq Q^{\sigma}_{\mathrm{tot}}(R) = Q^{\sigma}_{\max}(R)$
$R = Q^{\sigma}_{\mathrm{tot}}(R) \subsetneq Q^{\sigma}_{\max}(R)$
$Q^{\sigma}_{\mathrm{tot}} \subsetneq Q^{l}_{\mathrm{tot}},\ Q^{\sigma}_{\mathrm{tot}} \subsetneq Q^{r}_{\mathrm{tot}},$ $Q^{r}_{\mathrm{tot}} \neq Q^{l}_{\mathrm{tot}},\ Q^{r}_{\mathrm{tot}} \cong Q^{l}_{\mathrm{tot}}$

Ortega. Example with

$$\boxed{R \subsetneq Q^{\sigma}_{\mathrm{tot}}(R) \subsetneq Q^{\sigma}_{\max}(R)}$$

Open problems

1. Morita's construction of Q^r_{tot} can be adapted to construction of Q^σ_{tot}. However, an open question is if it ends after one step for R semihereditary as in right-sided case.
2. Following Stenström's question: Describe rings with $Q^\sigma_{\max} = Q^\sigma_{\mathrm{tot}}$.
3. Ortega – algorithms for calculating $Q^\sigma_{\max}$, $Q^r_{\max}$, $Q^l_{\max}$ for incidence and path algebras. For finite dimensional case $Q^\sigma_{\mathrm{tot}} = Q^\sigma_{\max}$. Find algorithms for calculating Q^σ_{tot}, Q^r_{tot} and Q^l_{tot} for infinite dimensional case.

References. Preprints of my papers are available on

http://www.usp.edu/∼lvas
and on arXiv.