Dimension of a class of Baer *-rings defined by a relaxed set of axioms

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Motivation

"Von Neumann algebras are blessed with an excess of structure – algebraic, geometric, topological – so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work."

"If all the functional analysis is stripped away ... what remains should (be) completely accessible through algebraic avenues".

Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

Goals

Let $\mathcal{A}$ be a finite von Neumann algebra.

- $\mathcal{A}$ has the **dimension function**:

$$\{ \text{ Modules over } \mathcal{A} \text{ } \} \to [0, \infty]$$

- Finitely generated $\mathcal{A}$-module **splits** as

torsion $\oplus$ finitely generated projective.

Goals :

1. To obtain an algebraic proof of these facts.
2. To generalize to a larger class $\mathcal{C}$ of Baer *-rings.

And:

3. To relax the axioms used to define $\mathcal{C}$.

From VNAs to Baer *-rings – Seven Reasonable Axioms

Berberian's Axioms:

(A1) A Baer *-ring R is **finite** if $x^*x = 1$ implies $xx^* = 1$ for all $x \in R$.

(A2) R satisfies **existence of projections** and **unique positive square root** axioms.

(A3) Partial isometries are addable.

(A4) R is **symmetric**: for all $x \in R$, $1 + x^*x$ is invertible.

(A5) There is a central element $i \in A$ such that $i^2 = -1$ and $i^* = -i$.

(A6) R satisfies the **unitary spectral** axiom.

(A7) R satisfies the **positive sum** -axiom.

What do we get from A1 – A7?

1. Berberian: R can be embedded in a **regular ring** Q satisfying A1–A7, having the same projections as R.

2. Vaš: R is **Ore** and $Q_{\text{cl}}(R) = Q = Q_{\max}(R)$.

3. Vaš: Ring of $n \times n$ matrices $M_n(R)$ over R is **semihereditary.**

4. Berberian: There is **dimension function**: $\{$ projection over R $\} \to$ con. functions on a nice space with values in $[0, \infty)$.

Need: $M_n(R)$ has complete lattice of projections for all n. Need two more axioms.

Two More Axioms – Class $\mathcal{C}$

(A8) $M_n(R)$ satisfies the **parallelogram law.**

(A9) Every sequence of orthogonal projections in $M_n(R)$ has a supremum.

Define the class $\mathcal{C}$ = class of Baer *-rings satisfying A1–A9. All the finite AW^*-algebras, finite VNA's are in $\mathcal{C}$.

With A8 and A9 we have:

- Berberian: $M_n(R)$ is a Baer *-ring with the dimension function on projections. $M_n(Q)$ is regular ring.
- Vaš:

  Finitely generated module =
  torsion $\oplus$ fin. gen. projective.

Defining the Dimension

Two steps:

1. If P is a fin. gen. projective R-module,

$$\dim_R(P) = d(p)$$

where p is a projection in $M_n(R)$ for some n, with $\operatorname{im} p \cong P$. The values are in $C_{[0,\infty)}(X)$.

2. If M is any R-module, define

$$\dim_R(M) = \sup \dim_R(P)$$

where sup is taken over fin. gen. proj. submodules of M. The values are in $C_{[0,\infty)}(X) \cup \{\infty\}$.

Properties of Dimension

Theorem [Vaš] $\dim_R$ has all the properties as the dimension of a finite von Neumann algebra.

1. **Extension**: two steps agree.
2. **Additivity** for short exact sequences.
3. **Cofinality**: dimension of directed union is supremum of dimensions.
4. **Continuity**: closure and dimension agree.
5. The dimension is **uniquely** determined by $1 - 4$.

Outline of the proof: Work over regular Q. Can go back to R since the projections are the same. Prove continuity using the monotony of f.g.p. modules.

Torsion Theories for $\mathcal{C}$

Define:

$\mathbf{T}$ = all modules of dimension zero and

$\mathbf{P}$ = dir. unions of fin. gen. projective modules.

Proposition [Vaš]

$(\mathbf{T}, \mathbf{P})$ is a torsion theory.

Theorem [Vaš]

$(\mathbf{T}, \mathbf{P})$ = Lambek = Goldie.

Every finitely generated module split as a direct sum of its torsion submodule and torsion-free quotient.

Torsion theories reflect ring theoretic properties of R

Denote:

- $(\mathbf{t}, \mathbf{p})$ = classical theory of an Ore ring R: M is torsion iff $M \otimes_R Q_{cl}(R) = 0$. For class $\mathcal{C}$, $\mathbf{p}$ = flats.

- $(\mathbf{b}, \mathbf{u})$ = largest theory in which R is torsion free.

Theorem [Vaš]

1. R is **regular** ($R = Q$) iff $(\mathbf{t}, \mathbf{p})$ is trivial.
2. If R is self-injective, then $(\mathbf{T}, \mathbf{P}) = (\mathbf{b}, \mathbf{u})$.
3. Q is **semisimple** iff $(\mathbf{t}, \mathbf{p}) = (\mathbf{T}, \mathbf{P})$ for R.
4. R is **semisimple** iff $(\mathbf{T}, \mathbf{P})$ is trivial.
5. R is **trivial** if and only if $(\mathbf{b}, \mathbf{u})$ is improper.

Going Back to Finite VNA

From this approach we get:

1. Reason why real valued dimension (dependent on the trace used) gives us Lambek = Goldie theory.

2. Algebra of affiliated operators has dimension and is Baer *-ring.

Getting rid of (A9)

Berberian: (A8) and (A9) are unwelcome guests.

Question: Can we get rid of them?

Theorem [Vaš]

If R is a Baer *-ring that satisfies (A1)–(A7), then $M_n(R)$ is Baer *-ring (thus, (A9) holds).

Proof follows from

1. Vaš: R is semihereditary and the regular ring Q of R is both the classical and the maximal ring of quotients.
2. M. W. Evans: The following are equivalent
   i) R is right semihereditary and $Q_{\max}(R)$ is the left and right flat epimorphic hull of R.
   ii) $M_n(R)$ is a right strongly Baer ring for all n.

Unwelcome Guest (A8)

Can we get rid of (A8) also? Berberian believes so. I agree. Problem is still open.

(A8) is used:

- In [Be] to show that $M_n(R)$ is finite.
- In [Va2] to show that the dimension function can be extended from projections in R to projections in $M_n(R)$.

References

[Be] Berberian, S. K. Baer $*$-rings; Springer-Verlag, Berlin-Heidelberg-New York, 1972.

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