Line and Surface Integrals. Stokes and Divergence Theorems

Review of Curves. Intuitively, we think of a curve as a path traced by a moving particle in space. Thus, a curve is a function of a parameter, say $t$. Using the standard vector representations of points in the three-dimensional space as $\mathbf{r} = (x, y, z)$, we can represent a curve as a vector function:

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

or using the parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$. The variable $t$ is called the parameter.

Example 1.

1. Line. A line in space is given by the equations

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

where $(x_0, y_0, z_0)$ is a point on the line and $(a, b, c)$ is a vector parallel to it. Note that in the vector form the equation $\mathbf{r} = \mathbf{r}(0) + \mathbf{m} t$ for $\mathbf{r}(0) = (x_0, y_0, z_0)$ and $\mathbf{m} = (a, b, c)$, has exactly the same form as the well known $y = b + mx$.

2. Circle in horizontal plane. Consider the parametric equations $x = a \cos t$ $y = a \sin t$ $z = b$. Recall that the parametric equation of a circle of radius $a$ centered in the origin of the $xy$-plane are $x = a \cos t$, $y = a \sin t$. Recall also that $z = b$ represents the horizontal plane passing $b$ in the $z$-axis.

Thus, the equations

$$x = a \cos t \quad y = a \sin t \quad z = b$$

represent the circle of radius $a$ in the horizontal plane passing $z = b$ on $z$-axis.

3. Ellipse in a plane. Consider the intersection of a cylinder and a plane. The intersection is an ellipse. For example, if we consider a cylinder with circular base $x = a \cos t$, $y = a \sin t$ and the equation of the plane is
\( mx + ny + kz = l \) with \( k \neq 0 \), the parametric equations of ellipse can be obtained by solving the equation of plane for \( z \) and using the equations for \( x \) and \( y \) to obtain the equation of \( z \) in parametric form. Thus \( z = \frac{1}{k}(l - mx - ny) \) and so \( x = a \cos t \ y = a \sin t \ z = \frac{1}{k}(l - ma \cos t - na \sin t) \).

4. **Circular helix.** A curve with equations \( x = a \cos t \ y = a \sin t \ z = bt \) is the curve spiraling around the cylinder with base circle \( x = a \cos t, y = a \sin t \).

5. **Plane curves.** All the concepts we develop for space curves correspond to plane curves simply considering that \( z = 0 \).

**Review of Line integrals of scalar functions.** Suppose that \( C \) is a curve given by \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) on the interval \( a \leq t \leq b \). Recall that the length of \( C \) is

\[
L = \int_C ds = \int_C |\mathbf{r}'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

This integral can be considered to be a special case of the situation when we integrate a scalar (real-valued) function \( f(x,y,z) \) over the curve \( C \). In the general case, we consider the line integral of \( C \) with respect to arc length as

\[
\int_C f(x,y,z) \, ds = \int_C f(\mathbf{r}(t)) \, |\mathbf{r}'(t)| \, dt = \int_a^b f(x(t),y(t),z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]

In Calculus 3, you may have seen the application of the this type of line integrals: finding the mass \( m \) and the center of mass \((\bar{x}, \bar{y}, \bar{z})\) of a wire \( C \) with density \( \rho(x,y,z) \). In particular, the mass can be calculated as

\[
m = \int_C \rho(x,y,z) \, ds.
\]

Using this example, you can think of the line integral as the total mass of the line density function over the curve \( C \).

**Example 2.** Evaluate the integral \( \int_C xy^3 \, ds \) where \( C \) is the circular helix \( x = 4 \cos t, y = 4 \sin t, z = 3t \), for \( 0 \leq t \leq \pi/2 \).

**Solution.** \( x' = 4 \cos t, y' = -4 \sin t, \ z' = 3 \Rightarrow ds = \sqrt{16 \cos^2 t + 16 \sin^2 t + 9} = \sqrt{25} = 5 \). Thus \( \int_C xy^3 \, ds = \int_0^{\pi/2} 4 \sin t \ 4^3 \cos^3 t \ 5 \, dt = (5)4^4 \underbrace{\frac{\cos t}{1}}_{10/2}^{\pi/2} = (5)4^3 = 320 \).

**Review of Line Integrals of vector functions.** Another type of line integrals includes integrating a vector function over a curve. Suppose now that \( \mathbf{f} \) is a function that assigns to each point
The following table summarizes the two types of line integrals.

**Line integrals of vector fields.** If \( \mathbf{f}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)) \) is a vector field, and the curve \( C \) is given by \( \mathbf{r}(t) = (x(t), y(t), z(t)) \), then the vector differential of the length element \( d\mathbf{r} \) is the product \( \mathbf{r}'(t)dt \). The line integral of \( \mathbf{f} \) along \( C \) is defined as

\[
\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C P \, dx + Q \, dy + R \, dz.
\]

This type of integrals measures the total effect of a given field along a given curve. In particular, many basic (non-continuous, one dimensional) formulas in physics such as \( s = vt \) can be represented in terms of line integrals in continuous and multi-dimensional cases, for example, \( s = \int v \, dt \).

Another example includes the formula for calculating the work done by the force \( \mathbf{F} \) (possibly an electric or gravitational field) in moving the particle along the curve \( C \)

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r}
\]

The length element \( ds \) and the vector differential of the length element \( d\mathbf{r} \) are related by

\[
d\mathbf{r} = \mathbf{r}'(t)dt \quad \text{and} \quad ds = |\mathbf{r}'(t)|dt.
\]

**Example 3.** Evaluate the integral \( \int_C x^2 \, dx + y \, dy + 2y \, dz \) where \( C \) consists of two parts \( C_1 \) and \( C_2 \). \( C_1 \) is the intersection of the cylinder \( x^2 + y^2 = 16 \) and the plane \( z = 3 \) from \((0,4,3)\) to \((-4,0,3)\). \( C_2 \) is a line segment from \((-4,0,3)\) to \((0,1,5)\).

**Solutions.** \( C_1 \) is on \( x^2 + y^2 = 16 \) thus \( x = 4 \cos t \) and \( y = 4 \sin t \). \( C_1 \) is also on \( z = 3 \) so

\[
x = 4 \cos t, \quad y = 4 \sin t, \quad z = 3
\]

are parametric equations of \( C_1 \). On \( C_1 \), \( dx = -4 \sin t \, dt, \ y = 4 \cos t \, dt \) and \( dz = 0 \). The point \((0,4,3)\) corresponds to \( t = \frac{\pi}{2} \) and the point \((-4,0,3)\) to \( t = \pi \). Thus, \( \int_{C_1} x^2 \, dx + y \, dy + 2y \, dz = \int_{\pi/2}^{\pi} 3^2(-4 \sin t)dt + 4 \sin 4 \cos t \, dt + 8 \sin t(0) = 36 \cos t + 8 \sin^2 t|_{\pi/2}^{\pi} = -36 - 8 = -44 .
\]

The line segment \( C_2 \) is passing \((-4,0,3)\) in the direction of the vector \( \overrightarrow{PQ} = (0,1,5) - (-4,0,3) = (4,1,2) \). So \( C_2 \) has equations \( x = -4 + 4t, \ y = t \) and \( z = 3 + 2t \) for \( 0 \leq t \leq 1 \). So, on this segment \( dx = 4 \, dt, \ dy = dt \) and \( dz = 2 \, dt \). \( \int_{C_2} x^2 \, dx + y \, dy + 2y \, dz = \int_0^1 (3 + 2t)^2 4 \, dt + 2 \, dt + 407 = 67.83.
\]

So, the final answer is \( \int_C = 67.83 - 44 = 23.83 \).

The following table summarizes the two types of line integrals.
After reviewing some basic facts about surfaces, we shall present analogous situation for surface integrals.

### Review of Surfaces

Adding one more independent variable to a vector function describing a curve \( x = x(t) \quad y = y(t) \quad z = z(t) \), we arrive to equations that describe a surface. Thus, a surface in space is a vector function of **two** variables:

\[
\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).
\]

These equations are called **parametric equations** of the surface and the surface given via parametric equations is called a **parametric surface**.

If \( x \) and \( y \) are used as parameters, the equations \( x = x, y = y, z = z(x, y) \) are frequently shortened to just \( z = z(x, y) \) and \( \mathbf{r}(x, y) = (x, y, z(x, y)) \) is also written shortly as \( z = z(x, y) \).

In some cases, a surface can be given by an **implicit function** \( F(x, y, z) = 0 \). In this case it is often needed to find parametric equations \( \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \). For example, a unit sphere can be described by \( x^2 + y^2 + z^2 = 1 \) can be parametrized as \( \mathbf{r} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \).

We recall the cylindrical and spherical coordinates which are frequently used to obtain parametric equations of some common surfaces.

**Cylindrical coordinates.**

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

Here \( x \) and \( y \) are converted using polar coordinates and the only change in \( z \) may come just from changes in \( x \) and \( y \). The Jacobian determinant can be computed to be \( J = r \). Thus, \( dx dy dz = r dr d\theta dz \).

**Spherical coordinates.** If \( P = (x, y, z) \) is a point in space and \( O \) denotes the origin, let

- \( r \) denote the length of the vector \( \overrightarrow{OP} = (x, y, z) \), i.e. the distance of the point \( P = (x, y, z) \) from the origin \( O \). Thus, \( x^2 + y^2 + z^2 = r^2 \).
• \( \theta \) be the angle between the projection of vector \( \overrightarrow{OP} = (x, y, z) \) on the \( xy \)-plane and the vector \( \vec{i} \) (positive \( x \) axis); and

• \( \phi \) be the angle between the vector \( \overrightarrow{OP} \) and the vector \( \vec{k} \) (positive \( z \)-axis). The conversion equations are

\[
x = r \cos \theta \sin \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \phi.
\]

The Jacobian determinant can be computed to be \( J = r^2 \sin \phi \). Thus, \( dx\,dy\,dz = r^2 \sin \phi \, dr\,d\phi\,d\theta \).

**Example 4.** The following are examples of parametric surfaces.

1. The cone \( z = \sqrt{x^2 + y^2} \) has representation using cylindrical coordinates as \( x = r \cos \theta, \ y = r \sin \theta, \ z = r \).

2. The paraboloid \( z = x^2 + y^2 \) has representation using cylindrical coordinates as \( x = r \cos \theta, \ y = r \sin \theta, \ z = r^2 \).

3. The sphere \( x^2 + y^2 + z^2 = 9 \) has representation using spherical coordinates as \( x = 3 \cos \theta \sin \phi, \ y = 3 \sin \theta \sin \phi, \ z = 3 \cos \phi \).

4. The cylinder \( x^2 + y^2 = 4 \) has representation using cylindrical coordinates as \( x = 2 \cos \theta, \ y = 2 \sin \theta, \ z = z \). The parameters here are \( \theta \) and \( z \).

5. The cylinder \( y^2 + z^2 = 4 \) has representation using cylindrical coordinates as \( x = x, \ y = 2 \cos \theta, \ z = 2 \sin \theta \). The parameters here are \( \theta \) and \( x \).
Cylinder $x^2 + y^2 = 4$

The Tangent Plane. For parametric surface $\mathbf{r} = (x(u, v), y(u, v), z(u, v))$, the derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ are vectors in the tangent plane. Thus, their cross product
\[
\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = (x_u, y_u, z_u) \times (x_v, y_v, z_v)
\]
is perpendicular to the tangent plane and, thus, to the surface as well.

If a surface is given by implicit function $F(x, y, z) = 0$, then this cross product also corresponds to the gradient $\nabla F$ of $F$.

Example 5. Find the equation of the tangent plane to a given surface at the specified point.
(a) $x^2 + 2y^2 + 3z^2 = 21$, $(4, -1, 1)$
(b) the cylinder $y^2 + z^2 = 4$, $(0, 3, 2)$.

Solutions. (a) The gradient is $(2x, 4y, 6z)$ and at the point $(4, -1, 1)$, the gradient is $(8, -4, 6)$. Thus the equation of the plane is $8(x - 4) - 4(y + 1) + 6(z - 1) = 0 \Rightarrow 8x - 4y + 6z = 42$.

(b) The cylinder can be parametrized as $x = 2\cos t$, $y = y$, $z = 2\sin t$. We find $(x_t, y_t, z_t) = (-2\sin t, 0, 2\cos t)$ and $(x_y, y_y, z_y) = (0, 1, 0)$. The cross product is $(-2\cos t, 0, 2\sin t)$. The $t$-value that corresponds to $(0, 3, 2)$ can be obtained from $x = 2\cos t = 0$ and $z = 2\sin t = 2$. Thus $t = \frac{\pi}{2}$ and plugging this value in the equation of the vector we obtained gives us $(0, 0, 2)$. So an equation of the plane can be obtained as $0(x - 0) + 0(y - 3) + 2(z - 2) = 0 \Rightarrow z = 2$.

Surface Integrals of scalar functions

Similarly as for line integrals, we can integrate a scalar or a vector function over a surface. Thus, we distinguish two types of surface integrals. The surface integrals of scalar functions are two-dimensional analogue of the line integrals of scalar functions.

- Line integral of a scalar function $\leftrightarrow$ Length
- Surface integral of a scalar function $\leftrightarrow$ Area

The surface area of the surface $\mathbf{r}(u, v)$ over the region $S$ in $uv$-plane can be obtained by integrating surface area elements $dS$ over sub-rectangles of region $S$. The area of each element $dS$ can be approximated with the area of the parallelogram in the tangent plane. The area of a parallelogram formed by two vectors is the length of their cross product.
Thus,

\[ dS = |r_u \times r_v| dudv \]

and so

\[
\text{Surface area} = \int \int_S dS = \int \int_S |r_u \times r_v| \, dudv
\]

This integral can be considered as a special case of the situation when we integrate the scalar function \( f = 1 \) over the surface \( r(u, v) \).

Let \( f(x, y, z) \) be a scalar (real-valued) function. Integrating \( f \) over the surface \( r(u, v) = (x(u, v), y(u, v), z(u, v)) \) we obtain the surface integral

\[
\int \int_S f(x, y, z) \, dS = \int \int_S f(r(u, v)) \, |r_u \times r_v| \, dudv
\]

In case when \( S \) is a region in \( xy \)-plane, and function \( f \) is a surface \( z = g(x, y) \), this integral represents the volume between \( z = g(x, y) \) and its projection on \( S \) in \( xy \)-plane. This is because \( r = (x, y, 0) \) in this case so \( r_x = (1, 0, 0), \ r_y = (0, 1, 0) \Rightarrow |r_x \times r_y| = 1 \).

The applications of the this type of line integrals include finding the mass \( m \) of a thin sheet \( S \) with the density function \( \rho(x, y, z) \). The mass \( m \) is given by

\[ m = \int \int_S \rho(x, y, z) \, dS \]

Using this example, you can think of the surface integral as the total mass of the surface density function over the surface \( S \).

In the special case where the surface is given via parameters \( x \) and \( y \) as \( z = z(x, y) \), \( dS = \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy \) so the surface integral of \( f \) over region \( S \) is

\[
\int \int_S f(x, y, z) \, dS = \int \int_S f(x, y, z(x, y)) \, \sqrt{1 + z_x^2 + z_y^2} \, dx \, dy
\]

Practice Problems.

1. Find the area of the following surfaces by using their parametric equations.

   (a) Part of \( z = y^2 + x^2 \) between the cylinders \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).
   (b) Part of the cone \( z = \sqrt{x^2 + y^2} \) between the cylinders \( x^2 + y^2 = 4 \) and \( x^2 + y^2 = 9 \).

2. Evaluate the surface integral where \( S \) is the given surface.
3. Use the parametrization of the previous problem so $dS = 4 \sin \phi d\theta d\phi$ again. The mass can be computed as $m = \int_S adS = a \int_0^{2\pi} \int_0^{\pi/2} 4 \sin \phi d\theta d\phi = 8a\pi$. 

**Solutions.**

1. (a) The paraboloid can be parametrized by $x = r \cos t, y = r \sin t, z = r^2$. Thus $r_t = (x_t, y_t, z_t) = (\cos t, \sin t, 2r), r_r = (x_r, y_r, z_r) = (-r \sin t, r \cos t, 0) \Rightarrow r_r \times r_t = (-2r^2 \cos t, -2r^2 \sin t, r) \Rightarrow |r_r \times r_t| = \sqrt{4r^4 \cos^2 t + 4r^4 \sin^2 t + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2 + 1)} = r \sqrt{4r^2 + 1}$.

The bounds for the integration are determined by the projection in the $xy$-plane which is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Thus $0 \leq t \leq 2\pi$ and $1 \leq r \leq 2$. So, the surface area is $S = \int_0^{2\pi} dt \int_1^2 r \sqrt{4r^2 + 1} dr = 2\pi 4.91 = 30.85$.

(b) The cone can be parametrized by $x = r \cos t, y = r \sin t, z = \sqrt{x^2 + y^2} = r$. $r_r = (\cos t, \sin t, 1)$ and $r_t = (-r \sin t, r \cos t, 0)$. $r_r \times r_t = (-r \cos t, -r \sin t, r)$. The length of this product is $\sqrt{r^2 \cos^2 t + r^2 \sin^2 t} + r^2 = \sqrt{r^2 + r^2} = 2r$.

The bounds for the integration are determined by the projection in the $xy$-plane which is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. Thus $0 \leq t \leq 2\pi$ and $2 \leq r \leq 3$. So, the surface area is $S = \int_0^{2\pi} dt \int_2^3 \sqrt{2rdr} = 2\pi \sqrt{2}(\frac{9}{3} - \frac{4}{3}) = 5\pi \sqrt{2}$.

2. (a) $dS = \sqrt{1 + z_x^2 + z_y^2} dxdy = \sqrt{1 + 4 + 9} dxdy = \sqrt{14} dxdy$. The integral is $\int_0^3 \int_0^2 x^2 y (1 + 2x + 3y) \sqrt{14} dxdy = \sqrt{14} \int_0^3 x^2 (10 + 4x) dx = 171 \sqrt{14}$

(b) $z = 1 - x - y \Rightarrow z_x = -1, z_y = -1$. $dS = \sqrt{1 + 1 + 1} dxdy = \sqrt{3} dxdy$. Bounds $0 \leq x \leq 1, 0 \leq y \leq 1 - x$. $\int \int_S yz dS = \int_0^1 \int_0^{1-x} y(1 - x - y) \sqrt{3} dxdy = \sqrt{3} \int_0^1 ((1 - x)\frac{y^2}{2} - \frac{y^3}{3})|_0^{1-x} dx = \sqrt{3} \int_0^1 \frac{(1-x)^3}{6} dx = \frac{\sqrt{3}}{2}$.

(c) $dS = \sqrt{1 + 1 + 1} dxdy = \sqrt{3} dxdy$. The integral is $\int \int y(y+3) \sqrt{2} dxdy = \int_0^{2\pi} \int_0^1 r \sin t (r \sin t + 3) \sqrt{2} r drdt = \sqrt{2} \int_0^{2\pi} \sin t (\frac{1}{4} \sin t + 1) = \sqrt{\frac{\pi}{2}}$. Alternatively, you can use the parametrization $x = r \cos t, y = r \sin t, z = r \sin t + 3$ with $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. Then $|r_r \times r_t| = r \sqrt{2}$ and the formula $\int_0^{2\pi} \int_0^1 r \sin t (r \sin t + 3) r \sqrt{2} drdt$ gives you the same answer.

(d) Recall that the sphere of radius 2 parametrizes by $x = 2 \cos \theta \sin \phi, y = 2 \sin \theta \sin \phi$ and $z = 2 \cos \phi$. Calculate $|r_\theta \times r_\phi|$ to be $4 \sin \phi$. The integral is $\int \int_S zdS = \int_0^{2\pi} \int_0^{\pi/2} 2 \cos \phi \sin \phi d\theta d\phi = 2\pi 8 \cdot \frac{1}{2} = 8\pi$.
Surface Integrals of Vector Fields. Flux

If \( r(u, v) \) is a surface, vector \( \mathbf{r}_u \times \mathbf{r}_v \) is perpendicular to the surface (i.e. the tangent plane). Considering the normalization of this vector, \( \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \), we arrive to the concept of the \textbf{unit normal vector} \( \mathbf{n} \).

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}
\]

If \( \mathbf{n} \) is a unit normal vector, then \( -\mathbf{n} \) is also a unit-length vector perpendicular to the surface, so both vectors \( \mathbf{n} \) and \( -\mathbf{n} \) can be used as unit normal vectors. Thus, we would need to be able to make a consistent choice of surface normal vector at every point. If that is possible, a surface is said to be \textbf{orientable} or \textbf{two-sided}. In this case, vector \( \mathbf{n} \) corresponds to unit normal vector of one side and \( n \) to the unit normal vector of the other side.

\textbf{Orientable and non-orientable surfaces.} Examples of orientable surfaces include planes, cylinders, and spheres.

A \textbf{Möbius strip} (or Möbius band) is an example of a surface that is not orientable. A model can be created by taking a paper strip and giving it a half-twist (180°-twists), and then joining the ends of the strip together to form a loop.

The Möbius strip has several curious properties: it is a surface with \textbf{only one side and only one boundary}. Convince yourself of these facts by creating your own Möbius strip or studying many animations on the web.

Another interesting property is that if you cut a Möbius strip along the center line, you will get one long strip with two full twists in it, not two separate strips. The resulting strip will have two sides and two boundaries. So, cutting created a second boundary. Continuing this construction you can deduce that a strip with an odd-number of half-twists will have only one surface and one boundary while a strip with an even-number of half-twists will have two surfaces and two boundaries.

There are many applications of Möbius strip in science, technology and everyday life. For example, Möbius strips have been used as conveyor belts (that last longer because the entire surface area of the belt gets the same amount of wear), fabric computer printer and typewriter ribbons. Medals often have a neck ribbon configured as a Möbius strip that allows the ribbon to fit comfortably around the neck while the medal lies flat on the chest. Examples of Möbius strip can be encountered: in physics as compact resonators and as superconductors with high transition temperature; in chemistry as molecular knots with special characteristics (e.g. chirality); in music theory as dyads and other areas.

For more curious properties and alternative construction of Möbius strip, see Wikipedia.

If a surface is two sided, \( \mathbf{n} \) corresponds to one side and \( -\mathbf{n} \) to the other. For a closed surface (i.e. compact without boundary), the convention is that the \textbf{positive orientation} is the one that corresponds to the normal vectors pointing outward and the \textbf{negative orientation} corresponds to the normal vectors pointing inward.
If the surface is not closed, the positive orientation can be defined by the **right hand rule**. Consider any closed, simple (i.e. does not cross itself nor it has missing points), smooth curve \( C \) on the surface and consider the positive (counter-clockwise) orientation on \( C \). The surface has the positive orientation if the normal vector \( n \) is always on the left of any vector parallel with it which is transversing the curve. That is: if you imagine yourself walking along \( C \) with your head pointing in the direction of \( n \), then the region \( S \) will always be on your left. Alternatively: if your index and middle fingers follow the direction of the curve, your thumb is pointing in the same direction as the vector \( n \).

If a surface is given by implicit equation \( F(x, y, z) = 0 \), the unit normal vector \( n \) can also be found as:

\[
 n = \frac{\nabla F}{|\nabla F|}.
\]

**Example 6.** Find the unit normal vector of the sphere \( x^2 + y^2 + z^2 = a^2 \).

**Solutions.** Consider \( F = x^2 + y^2 + z^2 - a^2 \) so that the gradient vector is \( \nabla F = (2x, 2y, 2z) \) and \( |\nabla F| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2a \). Thus, \( n = \frac{(2x, 2y, 2z)}{2a} = \frac{1}{a}(x, y, z) \).

**Flux integral.** If \( r(u, v) \) is an orientable surface with a tangent plane at every point, the vector differential of the surface area element \( dS \) can be considered to be the product of \( n \) and \( dS \) up to the sign. Thus,

\[
dS = n \ dS = \pm \frac{r_u \times r_v}{|r_u \times r_v|} \ |r_u \times r_v| \ dudv = \pm (r_u \times r_v) \ dudv.
\]

So, if \( f \) is a vector field \( f(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) \), the surface integral of \( f \) over region \( S \) on \( r(u, v) \) is given by

\[
\int \int_S \ f \cdot dS = \int \int_S \ f \cdot n \ dS = \pm \int \int_S \ f \cdot (r_u \times r_v) \ dudv
\]

The surface integral of a vector field is also called **flux integral**. The name comes from the fact that it computes the flux of fluid of density \( \rho \) and velocity field \( v \) flowing through surface region \( S \) when taking \( f \) to be the product \( \rho v \).

In fact, you can think of any flux integral of a vector function \( f \) as the **measure of the total flow of \( f \)** through the surface \( S \).

Besides fluid dynamics, this type of integral arises in other areas of physics. For example, if \( \vec{E} \) is an electric field, the surface integral of \( \vec{E} \) over the surface region \( S \) determines the electric flux of \( \vec{E} \) through \( S \). This integral is used to formulate the Gauss’ Law stating that the net charge enclosed by a closed surface region \( S \) is equal to the product of a constant \( \varepsilon_0 \) (the permittivity of free space) and the surface integral of \( \vec{E} \) over \( S \).

Another example of the use of this integral can be encountered in the study of heat flow. If \( K \) is a constant (called conductivity) and \( T \) is the temperature at point \( (x, y, z) \), the heat flow is defined
as $\vec{F} = -K\nabla T$ and the rate of heat flow across the surface $S$ is given by the surface integral of $\vec{F}$ over $S$.

The following table summarizes the relation of the two type of surface integrals.

| Surface integral of a scalar function $f(x, y, z)$ | $\int \int_{S} f(\mathbf{r}) \, dS = \int \int_{S} f(\mathbf{r}(u, v)) \left| \mathbf{r}_u \times \mathbf{r}_v \right| \, du \, dv$ |
|--------------------------------------------------|------------------------------------------------------------------|
| Surface integral of a vector function $\mathbf{f}(x, y, z)$ | $\int \int_{S} \mathbf{f}(\mathbf{r}) \cdot d\mathbf{S} = \pm \int \int_{S} \mathbf{f}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$ |

**Practice Problems.**

1. Find the flux integral of the vector field $\mathbf{f} = (y, x, z)$ over the part of the paraboloid $z = 1 - x^2 - y^2$ above the plane $z = 0$.

2. Find the flux integral of the vector field $\mathbf{f} = (y, x, z)$ over the boundary of the region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

3. Find the flux integral of the vector field $\mathbf{f} = (xze^y, -xze^y, z)$ over the part of the plane $x+y+z = 1$ in the first octant with the upward orientation.

4. Find the flux integral of the vector field $\mathbf{f} = (x, 2y, 3z)$ over the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

**Solutions.** (1) The flux integral is $\int \int_{S} (y, x, 1-x^2-y^2) \cdot (2x, 2y, 1) \, dxdy = \int \int_{S} (2xy+2xy+1-x^2-y^2) \, dxdy$. In polar coordinates, we have $\int_{0}^{2\pi} \int_{0}^{1} (4r^2 \sin t \cos t + 1 - r^2) r \, dr \, dt = \int_{0}^{2\pi} (\sin t \cos t + \frac{1}{4}) \, dt = \frac{\pi}{2}$.

(2) The flux integral is the sum of the integral of $\mathbf{f}$ over the paraboloid and the integral of $\mathbf{f}$ over the plane $z = 0$. The first integral is $\frac{\pi}{2}$ by the previous problem. The second integral is $\int \int_{S} (0, 0, -1) \, dxdy = \int \int_{S} 0 \, dxdy = 0$.

(3) The flux integral is $\int \int_{S} (xze^y, -xze^y, 1-x-y) \cdot (1, 1, 1) \, dxdy = \int_{0}^{1} \int_{0}^{1-x} (1-x-y) \, dxdy = \int_{0}^{1} (1-x-x(1-x) - \frac{1}{2}(1-x)^2) \, dx = \frac{1}{6}$.

(4) The cube consists of 6 surfaces. On top and bottom $z = \pm 1$ and $-1 \leq x, y \leq 1$. Those two flux integral yield $\int \int_{S} (x, 2y, \pm 3) \cdot (0, 0, \pm 1) \, dxdy = \int_{-1}^{1} \int_{-1}^{1} 3 \, dxdy = 12$.

On the left and right $y = \pm 1$ and $-1 \leq x, z \leq 1$. Those two flux integral yield $\int \int_{S} (x, \pm 2, 3z) \cdot (0, \pm 1, 0) \, dxdz = \int_{-1}^{1} \int_{-1}^{1} 2 \, dxdz = 8$.

On the front and back $x = \pm 1$ and $-1 \leq y, z \leq 1$. Those two flux integral yield $\int \int_{S} (\pm 1, 2y, 3z) \cdot (\pm 1, 0, 0) \, dydz = \int_{-1}^{1} \int_{-1}^{1} 1 \, dydz = 4$.

Thus, the total flux is $2(12+8+4)=48$. 

\[ \int_{-1}^{1} \int_{-1}^{1} 2 \, dxdy = 8 \]
Stokes’ Theorem

Stokes’ Theorem is a three-dimensional version of Green’s Theorem. Recall that Green’s theorem relates the line integral of a two-dimensional vector function $f = (P,Q)$ over a positive oriented, closed curve $C$ and the double integral over the interior $S$ of $C$.

$$\oint_C Pdx + Qdy = \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

In three-dimensional analogue, we relate the line integral of a three-dimensional vector function $f = (P,Q,R)$ over a closed curve $C$ and the surface integral of curl $\vec{f}$ over the interior of $C$. Recall that the curl of $f$ is defined as the vector product of $\nabla$ and $\vec{f}$.

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

Let $S$ be a region on an oriented piecewise-smooth surface $r(u,v)$ that is bounded by a simple, closed, piecewise smooth curve $C$. Recall that the orientation of $r$ induces the positive orientation of $C$ if the normal vector $\vec{n}$ of $r$ will always be on the left of any vector parallel with it that is transversing the curve (i.e. if you imagine yourself walking along $C$ with your head pointing in the direction of $\vec{n}$, then the region $S$ will always be on your left). In this case, if $f$ is a vector field, Stokes’ Theorem states that

$$\oint_C f \cdot dr = \iint_S \text{curl} f \cdot dS = \iint_S \text{curl} f \cdot \vec{n} \ dS$$

This relates to Green’s Theorem since if the curve $C$ is in $xy$-plane, $\vec{n} = (0,0,1)$, $\text{curl} f \cdot \vec{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, and $dS = \sqrt{0 + 0 + 1} dxdy = dxdy$ thus giving you the formula $\oint_C Pdx + Qdy = \iint_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$.

Stokes’ Theorem may be especially useful in the following cases:

1. The curl of $f$ is a simple function. In this case, evaluating the surface integral may be much easier that evaluation the line integral.

2. The curve $C$ consists of several pieces with different parametrization. In this case, it may be much faster evaluation the surface integral than several line integrals (one for each piece of $C$).

3. Assume that $S$ and $\bar{S}$ are regions on surfaces $r$ and $\bar{r}$ that have the same boundary $C$. Assume also that it is difficult to integrate over $S$. The Stokes’ theorem claims that we can use $\bar{S}$ instead since

$$\iint_S \text{curl} f \cdot dS = \oint_C f \cdot dr = \iint_{\bar{S}} \text{curl} f \cdot d\bar{S}$$
Practice Problems.

1. Evaluate \( \int_C \mathbf{f} \cdot d\mathbf{r} \) for \( \mathbf{f} = (x + y^2, y + z^2, z + x^2) \) and the curve \( C \) is the intersection of the plane \( x + y + z = 1 \) and the coordinate planes. (a) Without using Stokes’ Theorem; (b) Using Stokes’ Theorem.

2. Evaluate \( \int_C \mathbf{f} \cdot d\mathbf{r} \) for \( \mathbf{f} = (-y^2, x, z^2) \) and the curve \( C \) is the intersection of the plane \( y + z = 2 \) and the cylinder \( x^2 + y^2 = 1 \) oriented upwards. (a) Without using Stokes’ Theorem; (b) Using Stokes’ Theorem.

3. Show that the total work done by the force field \( \mathbf{f} = (yz, xz, xy) \) moving the particle along the intersection of the cylinder \( x^2 + y^2 = 1 \) and the sphere \( x^2 + y^2 + z^2 = 4 \) above the \( xy \)-plane is \( \int_C \mathbf{f} \cdot d\mathbf{r} = 0 \). When using Stokes’ Theorem, this problem becomes much shorter than without using it.

4. Find the work done by the force field \( \mathbf{f} = (x + z^2, y + x^2, z + y^2) \) when a particle moves under its influence around the edge of the part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies in the first octant oriented upwards.

Solutions. (1) (a) Without using Stokes’ Theorem: The curve \( C \) consists of three parts, \( C_1 \), \( C_2 \), and \( C_3 \) which are in the intersection of the plane and (1) the plane \( z = 6 \), (2) \( xz \)-plane, and (3) \( yz \)-plane, respectively. Positive orientation of \( C \) implies that \( C_1 \) is traversed from \((1,0,0)\) to \((0,1,0)\). On \( C_1 \) : \( x = x, y = 1 - x \) and \( z = 0 \) \( \Rightarrow \) \( dx = dx, dy = -dx \) and \( dz = 0 \) and the bounds are from 1 to 0. So, \( \int_{C_1} \mathbf{f} \cdot d\mathbf{r} = \int_{C_1} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_{1}^{0} (x + (1 - x)^2)dx + (1 - x)(-1)dx = \int_{1}^{0} x^2 dx = \frac{1}{3}. \)

On \( C_2 \) : \( x = 0, y = y, z = 1 - y \) \( \Rightarrow \) \( dx = 0, dy = dy \) and \( dz = -dy \) and the bounds are from 1 to 0. So, \( \int_{C_2} \mathbf{f} \cdot d\mathbf{r} = \int_{C_2} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_{1}^{0} (y + (1 - y)^2)dy + (1 - y)(-1)dy = \int_{1}^{0} y^2dy = \frac{1}{3}. \)

On \( C_3 \) : \( x = x, y = 0, z = 1 - x \) \( \Rightarrow \) \( dx = dx, dy = 0 \) and \( dz = -dx \) and the bounds are from 0 to 1. So, \( \int_{C_3} \mathbf{f} \cdot d\mathbf{r} = \int_{C_3} (x + y^2)dx + (y + z^2)dy + (z + x^2)dz = \int_{0}^{1} xdx + (1 - x + x^2)(-1)dx = \int_{0}^{1} (2x - 1 - x^2)dx = 1 - 1 = -\frac{1}{3}. \)

Thus \( \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -\frac{1}{3} + \frac{1}{3} - \frac{1}{3} = -1. \)

(b) With using Stokes’ theorem. Calculate that \( \text{curl} \mathbf{f} = (-2z, -2x, -2y). \) On the plane \( z = 1 - x - y, \) thus \( \mathbf{r} = (x, y, 1 - x - y) \) \( d\mathbf{S} = (1, 0, -1) \times (0, 1, -1) dx dy = (1, 1, 1) dx dy. \) Thus \( \int_S \text{curl} \mathbf{f} d\mathbf{S} = \int_{0}^{2\pi} \int_{0}^{1-x} (2z - 2x - 2y) dx dy = \int_{0}^{1-x} \left( -2 + 2x + 2y - 2x - 2y \right) dx dy = \int_{0}^{1-x} (1 - x) dx = -1. \)

(2) (a) \( C \) has parametrization \( x = \cos t, y = \sin t, z = 2 - y = 2 - \sin t, 0 \leq t \leq 2\pi. \) Thus \( \int_C \mathbf{f} \cdot d\mathbf{r} = \int_C -y^2 dx + xdy + z^2 dz = \int_{0}^{2\pi} \sin^3 t dt + \cos^2 t dt + (2 - \sin t)^3 \cos t dt = \pi. \)

(b) With Stokes: \( \text{curl} \mathbf{f} = (0, 0, 1 + 2y). \) \( d\mathbf{S} = (0, 1, 1) dx dy. \) \( \int_S \text{curl} \mathbf{f} d\mathbf{S} = \int_S (1 + 2y) dx dy = \int_{0}^{2\pi} \int_{0}^{1} (1 + 2\sin t) dr dt = \int_{0}^{2\pi} \left( \frac{1}{2} + 2 \sin t \right) dr dt = \pi. \)

(3) \( \text{curl} \mathbf{f} = 0. \) Thus \( 0 = \int_S \text{curl} \mathbf{f} d\mathbf{S} = \int_C \mathbf{f} \cdot d\mathbf{r}. \)

(4) It is easier to evaluate the integral using Stokes’ theorem (otherwise there would be three line integrals). Calculate \( \text{curl} \mathbf{f} \) to be \( (2y, 2z, 2x). \) The surface \( \mathbf{r} \) can be parametrized by \( x = 2 \cos \theta \sin \phi \) \( y = 2 \sin \theta \sin \phi \) \( z = 2 \cos \phi. \) Thus, \( d\mathbf{S} = \left( \frac{4 \sin^2 \phi \cos \theta}{2} \sin^2 \phi \cos \theta, \frac{4 \sin^2 \phi \sin \theta}{2} \cos \phi, 4 \sin \phi \cos \phi \right) d\phi d\theta. \) \( \int_S \text{curl} \mathbf{f} d\mathbf{S} = 16 \int^{\pi/2}_{0} \int^{\pi/2}_{0} (\sin^3 \phi \cos \theta \sin \sin^2 \phi \sin \theta, \sin \theta \cos \phi + \sin^2 \phi \cos \phi \cos \theta) d\phi d\theta = 16 \int_{0}^{\pi/2} \left( \frac{1}{2} \sin^3 \phi + 2 \sin^2 \phi \cos \phi \right) d\phi d\theta = 16. \)
Divergence Theorem

Recall that the divergence of \( \mathbf{f} \) is defined as the scalar product of \( \nabla \) and \( \vec{f} \).

\[
\text{div} \ \vec{f} = \nabla \cdot \vec{f} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

The Divergence Theorem can also be regarded as a three-dimensional version of Green’s Theorem in the form \( \oint_C P \, dy - Q \, dx = \iint_S \text{div} \mathbf{f} \, dx \, dy \). Here \( \mathbf{N} \) is the normal vector to \( C \) at point \((x, y)\) and the product \( \mathbf{N} \, ds \) can be calculated to be \( dy \vec{i} - dx \vec{j} \). This version of Green’s theorem relates the line integral of a two-dimensional vector function \( \mathbf{f} \) over a closed curve \( C \) with the double integral of \( \text{div} \mathbf{f} \) over the interior of \( C \). Adding one dimension to this formula, we relate the surface integral of a vector function \( \mathbf{f} \) with the triple integral of \( \text{div} \mathbf{f} \).

Let \( S \) be a region on a positive oriented surface that is the boundary of a simple solid region \( V \). If \( \mathbf{f} \) is a vector field, the Divergence Theorem states that

\[
\int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V \text{div} \mathbf{f} \, dx \, dy \, dz
\]

To summarize:

- **Green’s Theorem for** \( \mathbf{f} = (P, Q, 0) \) \( \Rightarrow \) \( \oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{f} \cdot k \, dx \, dy \)
- **Stokes’ Theorem for** \( \mathbf{f} = (P, Q, R) \) \( \Rightarrow \) \( \oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{f} \cdot \mathbf{n} \, dS \)
- **Green’s Theorem for** \( \mathbf{f} = (P, Q, 0) \) \( \Rightarrow \) \( \oint_C \mathbf{f} \cdot \mathbf{N} \, ds = \iint_S \text{div} \mathbf{f} \, dx \, dy \)
- **Divergence Theorem for** \( \mathbf{f} = (P, Q, R) \) \( \Rightarrow \) \( \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V \text{div} \mathbf{f} \, dx \, dy \, dz \)

**Practice Problems.**

1. Use the Divergence Theorem to find the flux of the vector field \( \mathbf{f} = (x, 2y, 3z) \) over the cube with vertices \((\pm1, \pm1, \pm1)\). (Note: the flux is found to be 48 without the use of the Divergence Theorem earlier.)

2. Find the flux of the vector field \( \mathbf{f} = (z, y, x) \) over the unit sphere.

3. Use the Divergence Theorem to find the flux of the vector field \( \mathbf{f} = (y, x, z) \) over the boundary of the region enclosed by the paraboloid \( z = 1 - x^2 - y^2 \) and the plane \( z = 0 \). (Note: the flux is found to be \( \frac{\pi}{2} \) without the use of the Divergence Theorem earlier.)

4. Find the flux of the vector field \( \mathbf{f} = (xy, yz, xz) \) over the boundary of the region enclosed by the cylinder \( x^2 + y^2 = 1 \), \( z = 0 \) and \( z = 1 \).

5. Find the flux of the vector field \( \mathbf{f} = (ye^z, y^2, xe^y) \) over the boundary of the region enclosed by the cylinder \( x^2 + y^2 = 9 \), \( z = 0 \) and \( z = y - 3 \).
6. Use the Divergence Theorem to find the flux of the vector field \( \mathbf{f} = (x, 2y, 3z) \) over the cube with vertices \((\pm1, \pm1, \pm1)\) without the top.

**Solutions.** (1) It is much easier to evaluate the integral using the Divergence theorem – recall that otherwise you have to do six flux integrals. \( \text{div} \mathbf{f} = 1 + 2 + 3 = 6 \) and so \( \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V 6 \, dx \, dy \, dz = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} 6 \, dx \, dy \, dz = 6(2)^3 = 48 \).

(2) It is easier to evaluate the integral using the Divergence theorem that directly. \( \text{div} \mathbf{f} = 1 \).
Thus, \( \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V 1 \, dx \, dy \, dz = \int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{1} r^2 \sin \phi \, dr \, d\phi \, d\theta = 2\pi(1 + 1)(1/2) = \frac{4\pi}{3} \).

(3) \( \text{div} \mathbf{f} = 1 \).
Thus, \( \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V 1 \, dx \, dy \, dz = \int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{1} r \, dr \, d\phi \, d\theta = 2\pi \int_{0}^{1} (1 - r^2) \, r \, dr \, d\theta = \frac{2\pi}{3} \).

(4) It is easier to evaluate the integral using the Divergence theorem than directly. \( \text{div} \mathbf{f} = y+z+x \).
Thus, \( \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V (x + y + z) \, dx \, dy \, dz = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} (r \cos t + r \sin t + z) \, r \, dz \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{1} (r \cos t + r \sin t + \frac{1}{3}) \, r \, dr \, d\theta = \frac{\pi}{2} \).

(5) It is easier to evaluate the integral using the Divergence theorem than directly. \( \text{div} \mathbf{f} = 2y \).
Thus, \( \int \int_S \mathbf{f} \cdot d\mathbf{S} = \int \int_V 2y \, dx \, dy \, dz \). The \( z \) bounds are determined from the two planes. Note that \( z = 0 \) is upper and \( z = y - 3 \) is lower so the integral becomes \( \int \int_V 2y \, dz \big|_{y-3}^{0} \, dx \, dy = \int \int V 2y(0 - y + 3) \, dx \, dy \) where the integrals are taken over the disk of radius 3. Using the polar coordinates, the integral reduces to \( \int_{0}^{2\pi} \int_{0}^{3} (-2r^3 \sin^2 t + 6r^2 \sin t) \, dr \, dt = \int_{0}^{2\pi} (-\frac{81}{2} \sin^2 t + 54 \sin t) \, dt = -\frac{81\pi}{2} \).

(6) Recall that \( \text{div} \mathbf{f} = 1 + 2 + 3 = 6 \) by problem 1. In order to use the Divergence Theorem, the top has to be considered. The flux over the top is \( \int \int_{\text{top}} (x, 2y, 3) \cdot (0, 0, 1) \, dx \, dy = \int_{-1}^{1} \int_{-1}^{1} 3 \, dx \, dy = 12 \).
Since

\[
\int \int_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} + \int \int_{\text{top}} \mathbf{f} \cdot d\mathbf{S} = \int \int_{V} 6 \, dx \, dy \, dz
\]

we have that

\[
\int \int_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} = \int \int_{V} 6 \, dx \, dy \, dz - \int \int_{\text{top}} \mathbf{f} \cdot d\mathbf{S}
\]

The triple integral was computed in problem (1) to be 48. Thus \( \int \int_{\text{no top}} \mathbf{f} \cdot d\mathbf{S} = \int \int_{V} 6 \, dx \, dy \, dz - \int \int_{\text{top}} \mathbf{f} \cdot d\mathbf{S} = 48 - 12 = 36 \).