Math 102
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Derivatives and Graphs. Graphical Analysis

1. Let
$$f(x) = \frac{1}{x} + \frac{x}{16}.$$
Find all possible maximum and minimum points and use the second derivative to determine whether each is a maximum or a minimum. Determine where the graph of f is concave up and where it is concave down.

2. Find the value of x at which
$$f(x) = \frac{\ln x + x}{x}$$
has a possible relative minimum or maximum point.

3. Let $f(x) = xe^{2x}$. Find all possible maximum and minimum points and use the second derivative to determine whether each is a maximum or a minimum. Determine where the graph of f is concave up and where it is concave down.

4. Suppose that $f(x)$ is a function with the following derivatives
$$f'(x) = \frac{(x-8)(x+1)}{(x+4)}, \qquad f''(x) = \frac{(x+10)(x-2)}{(x+4)^2}$$
Find the critical points of f. Determine the intervals on which $f(x)$ increase and the intervals on which f decrease. Use the First or the Second derivative test to check if there is a maximum or a minimum at $x = -1$ and $x = 8$. Determine the intervals on which $f(x)$ is concave upward and the intervals on which $f(x)$ is concave downward.

5. Sketch the graph of a function having the given properties
$$f(0) = 1; \quad f'(0) = 0; \quad f''(x) > 0, \text{ for all values of } x$$
What is the extreme value of f? Is it a minimum or maximum? What can you say about concavity of f?

6. Sketch the graph of a function having the properties:
$$f(3) = 1; \quad f(-3) = -1; \quad f(0) = 0;$$
$$f'(3) = 0; \quad f'(-3) = 0;$$
$$f''(x) > 0 \text{ for } x < 0, \text{ and } f''(x) < 0 \text{ for } x > 0.$$
What are the extreme values of f? What kind of extreme values are they? What can you say about concavity of f? What is the inflection point of f?

7. Sketch the graph of a function having the properties:

$$f(-2) = -1; \quad f(2) = -1; \quad f(0) = 1;$$
$$f'(-2) = 0; \quad f'(2) = 0; \quad f'(0) = 0;$$
$$f"(x) > 0 \text{ for } x < -1 \text{ and } x > 1;$$
$$\text{and } f"(x) < 0 \text{ for } -1 < x < 1.$$

What are the extreme values of f? What kind of extreme values are they? What can you say about concavity of f? What are the inflection points of f?

8. Suppose that a function f satisfies the following conditions:

$$f(-2) = 2; \quad f(0) = -2; \quad f \text{ has a vertical asymptote at } x = 2$$
$$f'(-2) = 0; \quad f'(0) = 0; \quad f' \text{ is not changing sign at } x = 2$$
$$f''(x) < 0 \text{ for } x < -1 \text{ and } x > 2;$$
$$\text{and } f''(x) > 0 \text{ for } -1 < x < 2.$$

What are the critical points of f (if any)? What kind of extreme values are they (if any)? On which intervals is f concave upwards? On which intervals is f concave downwards? What are the inflection points of f (if any)? Sketch the graph of f.

Solutions:

1. f is increasing for $x < -4$ and $x > 4$; decreasing for $-4 < x < 0$ and $0 < x < 4$; f is concave upwards for $x > 0$; concave downward for $x < 0$; the relative minimum is (4, 1/2); the relative maximum is (-4, -1/2); there are no inflection points.

2. f is increasing for $0 < x < e$; decreasing for $x > e$; f is concave upwards for $x > e^{3/2}$; concave downward for $0 < x < e^{3/2}$; there is no relative minimum; the relative maximum is $(e, (1 + e)/e) = (2.72, 1.37)$; the inflection point is $(e^{3/2},) = (4.48, 1.33)$.

3. f is increasing for $x > -1/2$; decreasing for $x < -1/2$; f is concave upwards for $x > -1$; concave downward for $x < -1$; the relative minimum is $(-1/2, -.18)$; there is no relative maximum; the inflection point is $(-1, -.135)$.

4. f is increasing for $x > 8$ and $-4 < x < -1$; decreasing for $x < -4$ and $-1 < x < 8$. At $x = -1$ there is a maximum; at $x = 8$ there is a minimum. f is concave upwards for $x > 2$ and $x < -10$; concave downward for $-10 < x < 2$.

5. (0,1) is a relative minimum. f is concave upwards for all values of x.

6. (3,1) is a relative maximum, (-3, -1) is a relative minimum. f is concave upward for $x < 0$, f is concave downward for $x > 0$. (0,0) is an inflection point.

7. (-2, -1) and (2, -1) are relative minima, (0,1) is a relative maximum. f is concave up for $x < -1$ and $x > 1$. f is concave down for $-1 < x < 1$. f has inflection points at $x = 1$ and $x = -1$.

8. Critical points: -2, 0 and 2. (-2, 2) is a relative maximum, (0,-2) is a relative minimum. $x = 2$ is not an extreme point. f is concave up for $-1 < x < 2$. f is concave down for $x < -1$ and $x > 2$. f has inflection points at $x = -1$.