MATH 102
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Review for the Final Exam

1. **Limits.** Evaluate the following limits.

a) $\lim_{x\to 3} \dfrac{x^2-x-6}{x^2-2x-3}$

b) $\lim_{x\to\infty} \dfrac{x^2-x-6}{x^2-2x-3}$

c) $\lim_{x\to-\infty} 5^x+3$

d) $\lim_{x\to\infty} 5^x+3$

e) $\lim_{x\to 3^-} \dfrac{2}{x-3}$

f) $\lim_{x\to 3} \dfrac{2}{x-3}$

g) $\lim_{x\to\infty} \dfrac{2}{x-3}$

h) $\lim_{x\to\infty} \dfrac{2x}{x-3}$

i) Let

$$f(x) = \begin{cases} -x-1 & x<-1 \\ 1-x & -1\leq x<1 \\ \sqrt{x-1} & x\geq 1 \end{cases}$$

Evaluate the following:

$\lim_{x\to-1^-} f(x)$ $\qquad \lim_{x\to-1^+} f(x)$ $\qquad \lim_{x\to-1} f(x)$ $\qquad f(-1)$ $\qquad \lim_{x\to 1} f(x)$ $\qquad \lim_{x\to 0} f(x)$

j) Let

$$f(x) = \begin{cases} (x+1)^2 & x\leq -1 \\ x+2 & -1<x<2 \\ -2x+8 & x\geq 2 \end{cases}$$

Evaluate the following:

$\lim_{x\to-1^-} f(x)$ $\qquad \lim_{x\to-1^+} f(x)$ $\qquad \lim_{x\to-1} f(x)$ $\qquad f(-1)$ $\qquad \lim_{x\to 2^-} f(x)$ $\qquad \lim_{x\to 2} f(x)$

2. **Derivatives.** Find the derivative for the given function.

a) $y=\frac{4}{x^2}-\frac{1}{3x^6}$

b) $y=(x-\frac{1}{x})(\sqrt[3]{x}+5x^2)$

c) $y=\frac{\sqrt{x}+x}{x^2+1}$

d) $y = \frac{(3x^2+1)(x+4)}{5-x^2}$

e) $y = \frac{\sqrt{x}(2x^3-5)}{3x+2}$

f) $f(x) = \ln(5x - e^{5x})$

g) $f(x) = \log_2(x^2 + 7x)$

h) $f(x) = \frac{(x^2+3)^4}{(3x^2+1)^5}$

i) $f(x) = \sqrt{\frac{x^2-1}{x^2+2}}$

j) $f(x) = e^{3x}(x^3 + 2x - 5)$

k) $f(x) = 3^{2x^2+5}$

l) $f(x) = x\ 5^{3x}$

m) $f(x) = \sin(2x^2 + 4)$

n) $f(x) = x^2 \cos(x^2)$

o) $f(x) = \sin 3x \cos 5x$

3. **Implicit Differentiation.** Find the derivative $\frac{dy}{dx}$.

   a) $x^2 + xy = 6$

   b) $x^3 + 12xy = y^3$

   c) $xe^y + x^2 = y^2$

4. **Average and instantaneous rate of change.**

   a) Let $f(x) = \sqrt{x}(x^2 - 10)$. Find the average rate of change over $[0, 4]$. Find the instantaneous rate of change at $x = 4$.

   b) Let $f(x) = \frac{x^2+4}{x}$. Find the average rate of change over $[1, 2]$. Find the instantaneous rate of change at $x = 1$.

5. **Tangent Line.** Find an equation of the line tangent to the graph of the given equation at the indicated point.

   a) $f(x) = \sqrt{x^3} + \sqrt[3]{x^2}$; $(1, 2)$

   b) $y = \frac{e^{2x}-1}{e^{2x}+1}$; $(0, 0)$

   c) $y = \ln\sqrt{2x-1}$; $(1, 0)$

   d) $x^2 + y^2 = 13$; $(3, 2)$

   e) $x \ln y = 2x^3 - 2y$; $(1, 1)$

   f) $x^2 + y^2 = e^y$; $(1, 0)$

6. **Linear Approximations.**

   a) If $f(2) = 5$ and $f'(2) = 3$, approximate $f(2.1)$.

   b) If $f(2) = 5$ and $f'(2) = 3$, approximate $f(1.9)$.

c) Use the linear approximation to estimate $\sqrt[3]{26}$. Compare to the calculator value of $\sqrt[3]{26}$.

d) Use the linear approximation to estimate $\sqrt[4]{16.2}$.

7. **Applications of Derivatives**

a) A company determines that its cost function is $C(x) = 1000 + 35x - .01x^2$, $0 \le x \le 300$, where x is the number of items produced and $C(x)$ is the cost of producing x items in dollars. Find the average rate of change in cost when x is changing from 100 to 150. Then, find the instantaneous rate of change in cost when producing 200 units and estimate the cost of producing 201 items.

b) Assume that the mathematical model for the growth of a locust tree in its first century of life is given by $h(t) = 3\sqrt{t}$, $0 \le t \le 100$, where t is the age of the tree in years and $h(t)$ is the height of the tree in feet. Find and interpret $h(64)$ and $h'(64)$.

c) The size of bacteria culture at time t in minutes, is approximated by $N(t) = 4t^{7/2}$, in milligrams. Find and interpret $N(9)$ and $N'(9)$. Using the linear approximation, estimate $N(9.2)$.

d) If a stone is dropped from a building 150 feet tall, its height above the ground after t seconds is given by $s(t) = 150 - 16t^2$, in feet. Find the average velocity of the stone between 1 and 3 seconds after it is dropped. Find the velocity 2.5 seconds after the stone is dropped.

e) The percent concentration of a certain medication during the first 20 hours after it has been administered is approximated by $p(t) = \frac{230t}{t^2+6t+9} \quad 0 \le t \le 20$ Determine at which hour is the concentration maximal. What is the maximal concentration?

f) A company determines that its revenue function is $R(x) = 15.22xe^{-.015x}$. What production level produces the maximal revenue? What is the maximal revenue?

8. **Related Rates.**

a) Suppose a spherical balloon is inflated at the rate of 10 cubic centimeters per minute. How fast is the radius of the balloon increasing at the time when the radius is 5 cm? Recall that the formula for the volume of a sphere is $V = 4/3\ \pi r^3$.

b) Water leaking onto a floor creates a circular puddle with an area that increases at the rate of 3 square centimeters per minute. How fast is the radius of the puddle increasing when the radius is 10 cm? Recall that the formula for the area of a circle is $A = r^2\pi$.

c) Assume that the number of bass in the pond is related to the level of polychlorinated biphenyls (PCBs, a group of industrial chemicals) in the pond. The bass population is modeled by $y = \frac{2500}{1+x}$ where x represents the PCB level in parts per million (ppm) and y represents the number of bass in the pond. If the level of PCBs is increasing at the rate of 40 ppm per year, find the rate at which is the number of bass changing when there are 100 bass in the pond.

d) Water is leaking onto a floor from a dripping faucet. The water is forming a semi-circular puddle next to a wall. The puddle is growing in surface area at a rate of 10 square inches per minute. How rapidly is the radius of the puddle growing at the moment when the area of the puddle is 50π square inches?

9. **Derivatives and Graphs.** Graph the given function. Choose the appropriate scale to see the entire graph with all the relevant points (intercepts, extreme and inflection points) on it. Find the intervals where the function is increasing and where it is decreasing. Find the intervals where the function is concave upward and where it is concave downward. Find the relative minimum, relative maximum and the inflection points.

   a) $f(x) = x^3/3 + x^2 - 15x + 3$
   b) $f(x) = \frac{1}{x} + \frac{x}{16}$
   c) $f(x) = \frac{\ln x + x}{x}$
   d) $f(x) = xe^{2x}$

10. **Graphical Analysis.**

    a) Sketch the graph of a function having the given properties

    $$f(0) = 1; \quad f'(0) = 0; \quad f"(x) > 0, \text{ for all values of } x$$

    What is the extreme value of f? Is it a minimum or maximum? What can you say about concavity of f?

    b) Sketch the graph of a function having the properties:

    $$f(3) = 1; \quad f(-3) = -1; \quad f(0) = 0;$$

    $$f'(3) = 0; \quad f'(-3) = 0;$$

    $$f"(x) > 0 \text{ for } x < 0, \text{ and } f"(x) < 0 \text{ for } x > 0.$$

    What are the extreme values of f? What kind of extreme values are they? What can you say about concavity of f? What is the inflection point of f?

11. **Optimizing Functions.** Find the absolute minimum and maximum of each function on the indicated interval. You can use your calculator to find the zeroes of the first derivative if necessary.

    a) $f(x) = 3x^4 + 4x^3 - 36x^2 + 1; \quad [-1, 4]$
    b) $f(x) = x^4 - 3x^3 - 8x^2 + 12x + 16; \quad [1, 4]$

12. **Optimization Problems.**

    a) Find the dimensions of the rectangular garden of greatest area that can be fenced off with 400 feet of fencing.

b) Consider a box with a square base. Find the dimensions of the box with the surface area 96 square inches, such that the volume is as large as possible.

c) The Hardy - Weinberg Law states that the proportion of individuals in a population who are heterozygous is $2pq$ and the proportion of individuals who are homozygous is $p^2 + q^2$. Recall that $p + q = 1$. a) Find the maximal percentage of people that are heterozygous. b) Find the minimal percentage of people that are heterozygous. c) Find the minimal percentage of people that are homozygous. d) Find the maximal percentage of people that are homozygous.

d) The concentration of a certain medication in a patient's bloodstream can be given by $C(t) = \frac{5.3t}{t^2+4t+5}$ $\quad 0 \leq t \leq 8$ where $C(t)$ is in milligrams per cubic centimeter and t is the number of hours after the medication has been administered. How many hours after the medication has been administered is the concentration at a maximum? What is the maximum concentration?

e) An open top box is made with a square base and should have a volume of 6000 cubic inches. If the material for the sides costs \$.20 per square inch and the material for the base costs \$.30 per square inch, determine the dimensions of the box that minimize the cost of the materials.

13. **Definite and Indefinite Integrals.** Evaluate the following integrals.

a) $\int (\sqrt{x} - \frac{4}{x^2})\, dx$

b) $\int_1^4 (\frac{3}{\sqrt{x}} - 2x)\, dx$

c) $\int (3x + 5)^6\, dx$

d) $\int_0^4 (3x - 5)^2\, dx$

e) $\int (\frac{1}{x} - \frac{1}{x^2})\, dx$

f) $\int \frac{1}{4x+1}\, dx$

g) $\int \frac{x^2}{x^3+1}\, dx$

h) $\int (e^{2x} + e^{-2x})\, dx$

i) $\int_0^1 3^{2x}\, dx$

j) $\int \cos(5x + 1)\, dx$

k) $\int_0^{\pi/2} \sin(2x)\, dx$

14. **Approximate Integration.** Approximate the following integral using the Left and Right Sums Program to first two nonzero digits.

a) $\int_0^2 \ln(x^2 + 1)\, dx$

b) $\int_1^3 \frac{e^{2x}}{x}\, dx$

15. **Area.** Find the following areas.

a) Area between $f(x) = x^2 - 2x$ and x-axis for $0 < x < 2$

b) Area between $f(x) = x^2 - 2x$ and x-axis for $1 < x < 3$

c) Area between $f(x) = 2\sqrt{x} - 4$ and x-axis for $0 < x < 9$

d) Area between $f(x) = 2/x$ and $g(x) = 4/x^2$ for $1 < x < 4$

e) Area between $f(x) = 4 - x^2$ and $g(x) = -x + 2$

f) Area between $f(x) = x^3$ and $g(x) = 3x^2 - 2x$

16. **Applications of Integrals.**

a) The size of a certain bacteria culture grows at a rate of $f(t) = te^{t/2}$ milligrams per hour. Use your calculator program to approximate the bacteria size after the first 3 hours to first two nonzero digits.

b) From past records, a botanist knows that a certain species of tree has a rate of growth that can be modeled by $f(x) = 2/\sqrt{x}$, $1 \leq x \leq 4$, where x is the age of the tree in years and $f(x)$ is the growth rate in feet per year. Determine how much did the tree grow from the time when it was a year old to the time it was four years old.

c) Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{\sqrt{t^2+9}}$ where t is the time in seconds and v is the velocity in feet per second.
a) Determine the total movement of the object between 3 and 5 seconds.
b) Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.

d) Geologists estimate that an oil field will produce oil at a rate given by $f(t) = 600e^{-0.1t}$ thousand barrels per month, t months into production. Estimate the total production for the first year of operation. Round to the nearest whole number.