MATH 101
Lia Vas

Modeling with Exponential and Logarithmic Functions

Recall that a linear function has constant average rate of change: difference of two y-values for equally spaced x-values is constant.

Exponential function has constant percent (relative) rate of change: quotient of two y-values for equally spaced x-values is constant.

Exponential function:

$$P = P_0 a^t$$

where P_0 is the initial quantity, t is independent variable (e.g. time), and $a = 1 + r$ is the growth ($a > 1$) or decay ($0 < a < 1$) factor and r is the discrete (e.g. annual) percent rate of change. For example, this formula is used to model the population growth with given annual percent rate given.

If k is the continuous percent rate of change, then the function

$$P = P_0 e^{kt}$$

models the change of desired quantity. Connection: if $k = \ln(1 + r) \approx r$ if r is small.

If k is positive it is called the **growth factor**. If k is negative, then the function is decreasing and k is called the **decay factor**. For example, this formula is used to model the decay of a radioactive substance.

Practice Problems.

1. A bacteria culture grows by the exponential model $y = 200e^{kt}$. How many bacteria are there initially? If the number of bacteria triples in 2 hours, find the number of bacteria after 5 hours.

2. A bacteria culture starts with 600 bacteria and grows by the exponential model $y = y_0 e^{kt}$. After 3 hours there are 2400 bacteria. Find the number of bacteria after 4 hours. When will the number of bacteria be 5000?

3. Suppose that a population grows by 3% per year. Find the time it would take for the population to double.

4. Suppose that a population grows by 5% per year. Find the time it would take for the population to triple.

5. Happyville and Smileytown both have a population of 10,000 people presently. Happyville is increasing by 1500 people a year and Smileytown is increasing by 15% a year.

   a) Which town is growing faster?

b) Find formulas for the populations of these towns as function of time t in years.

c) Use part b) to predict the size of both towns 5 years from now.

d) Find a year in which population of Happyville will be over 25000. Do the same for Smileytown.

6. The half-life of bismuth 210 is 5 days. How many days it will take the 1.5 grams of bismuth 210 to decay to 0.3 grams?

7. The biological half-life of triazolam, a drug used for treating insomnia, is 2.3 hours. What percent of an initial dose will remain after 5 hours?

8. A radioactive substance decays to a half of the initial amount in 8 years. There are 200 grams present initially. Give a possible formula for the amount of this substance as a function of time t in years. How long will it take for the substance to decay to 10%of the original amount?

9. The time T in minutes for a small plane to climb to an altitude of h feet is given by $T(h) = 50\log(\frac{20000}{20000-h})$ where 20000 is the plane's absolute ceiling. How long would it take the plane to climb to an altitude of 5000 feet? Can the plane ever reach the absolute ceiling? How high is the plane 50 minutes after the take of?

10. The amount of the drug (in milligrams) present in patients bloodstream t hours after it has been administered can be modeled by $A(t) = 5.5e^{-.234t}$, How many milligrams are initially administered? After how many hours will there be 1 mg of the drug in the bloodstream?

11. The pH of a liquid is defined as $pH = -\log[H^+]$, where $[H^+]$ is the hydrogen ion concentration in moles per liter. Find the hydrogen ion concentration for a pH of 5.2. If $9.98 \cdot 10^{-8}$ is the the hydrogen ion concentration for blood, find the pH.

12. Suppose that the spread of a flu at a school with 2000 students total is given by $f(t) = \frac{2000}{1+1999e^{-.8t}}$ where t represent the number of days and f represents the number of students infected. How many students are initially infected? How many students are infected after a week? After how many days will 45% of the total number of students be infected?

Solutions:

1. 200 bacteria. 3118 bacteria.
2. 3810 bacteria. 5000 bacteria in 4.6 hours.
3. $P = P_0(1.03)^t$. $t = 23.45$ years.
4. $P = P_0(1.05)^t$. $t = 22.52$ years.
5. a) Smileytown; b) $y_1 = 10000+1500t, y_2 = 10000(1.15)^t$; c) 17500 and 20114 people; d) $t = 10$ years and $t = 6.556 \approx 6.5$ years.
6. 11.6 days
7. 22.16%
8. $y = 200e^{-.0866t}$. 26.58 years.
9. 6.25 minutes. No (zero in the denominator). 18000 feet.
10. 5.5 milligrams. 7.3 hours
11. $6.31 \cdot 10^{-6}$ moles per liter. pH is 7.4.
12. Initially, 1 student; week later, 238; 9.25 days after, 45% of 2000 are infected.