MATH 101
Lia Vas

Geometric Series

A **series** is a sum of numbers indexed by the positive (or nonnegative) integers:

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

For example, $1 + 1/2 + 1/3 + 1/4 + \ldots$

A series $a_1+a_2+a_3+\ldots$ is **convergent** if there is a number a such that $a_1+a_2+a_3+\ldots = a$. Otherwise, it is **divergent.**

The sum $s_n = a_1+a_2+a_3+\ldots+a_n$ of the first n terms of the series $a_1+a_2+a_3+\ldots+a_n+\ldots$ is called **n-th partial sum.**

The **geometric series** of radius r is the series $a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$. The n-th partial sum for the geometric series (with $r \neq 1$) is

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

The geometric series is convergent for $-1 < r < 1$ and has the sum

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}.$$

Problems.

1. Find the sum of the following convergent geometric series.
   a) $1 + 1/2 + 1/2^2 + 1/2^3 + \ldots$
   b) $3 + 3/2 + 3/2^2 + 3/2^3 + \ldots$
   c) $2 - 2/3 + 2/9 - 2/27 + 2/81 - \ldots$
   d) $4 + 4/10 + 4/100 + 4/1000 + \ldots$
   e) $35/100 + 35/10000 + 35/1000000 + \ldots$
2. Represent the following decimal number as a quotient of integers.
   a) $0.222222\ldots$ b) $0.27272727\ldots$

Solutions.

1. a) sum=2; b) sum=6; c) sum=3/2; d) 4.4444444...=40/9; e) sum=.353535...=35/99.
2. a) 2/9 b) 3.11

Applications of Geometric Series

If a person is given the same dose of a medicine at equally spaced time intervals, the body metabolizes some of the drug so that, after some time, only a certain percent of the original amount remains. After the each dose, the amount of the drug in the body is equal to the amount of the given dose plus the amount remnant from the previous doses. Let a be the amount given in each dose, and r the percent of the previous dose remaining in the body. Then, the amount A_n of the drug present after the n-th dose is

$$A_n = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

After a long time, the amount of the drug in the body present before and after the new dose stabilizes. The amount of the drug present after the new dose stabilizes at

$$A = a + ar + ar^2 + \ldots \frac{a}{1-r}.$$

The amount A is called the **steady state**. The amount of the drug present before the new dose stabilizes at $A - a$.

The similar analysis can be used to compute the amount of herbicides or pesticides accumulated in people's bodies, how long the natural resources will last assuming that the current usage levels increase at a constant rate and other phenomena.

Note that different analysis is needed if the drug is not given in discrete doses but **continuously** (such as intravenously). If the rate at which the drug is eliminated is continuous and equal to the percent p, the radius of the geometric series r is e^{-p}.

In business and economy, geometric series are used for computing the amount present in an account if the deposits are made repeatedly, annuities, market stabilization point etc.

Problems.

1. A person with an ear infection takes 200 mg ampicillin tablet once every 4 hours. About 12% of the drug in the body at the start of a four hour period is still there at the end of that period. What quantity of ampicillin is in the body

   a) Right after taking the third tablet? b) Right after taking the sixth tablet?

   c) At the steady state level right after taking a tablet?

   d) At the steady state level right before taking a tablet?

2. A person takes 100 mg of a drug at regular time intervals. About 15% of the drug in the body at the start of a new time period is still there at the end of that period. What quantity of the drug is in the body a) right after taking the fourth dose; b) in the long run right after taking a dose; c) in the long run right before taking a dose?

3. Every day person consumes 5 micrograms of a toxin which leaves the body at a rate of 2% per day. How much toxin is accumulated in the body in the long run?

Solutions.

1. a) 226.88 mg b) 227.27 mg c) 227.27 mg; d) 27.27 mg.

2. a) 117.59 mg b) 117.65 mg c) 17.65 mg.

3. Note $r = .98$ here. Get $A = 250$ micrograms. If using continuous model, $r = e^{-.02}$ get $A = 252.5$ micrograms in this case. Note that continuous model is more realistic in this case.